BERGMAN-HARMONIC FUNCTIONS ON CLASSICAL DOMAINS

MING XIAO*

Abstract. We study Bergman-harmonic functions on classical domains from a new point of view in this paper. We first establish a boundary pluriharmonicity result for Bergman-harmonic functions on classical domains: A Bergman-harmonic function u on a classical domain D must be pluriharmonic on germs of complex manifolds in the boundary of D if u has some appropriate boundary regularity. Next we give a new charaterization of pluriharmonicity on classical domains which may shed a new light on future study of Bergman-harmonic functions. We also prove characterization results for Bergman-harmonic functions on type I domains.

2010 Mathematics Subject Classification: Primary 32A50 Secondary 32M15

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ equipped with the Bergman metric $g = (g_{ij})_{1 \leq i,j \leq n}$. The Laplace-Beltrami operator with respect to the Bergman metric on Ω is defined as

$$\Delta_g = 4 \sum_{1 \leq i,j \leq n} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \overline{z}_j}.$$

Here $(g^{i\bar{j}})$ is the inverse matrix of $(g_{i\bar{j}})^t$. For instance when Ω is the unit ball $\mathbb{B}^n$, Δ_g is given by the following formula up to a constant multiplication (cf. [H]):

$$(1 - ||z||^2) \sum_{\alpha,\beta=1}^{n} (\delta_{\alpha\beta} - z_\alpha \overline{z}_\beta) \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta}.$$

We will follow the convention of Graham [G1-2] and say a function $u \in C^2(\Omega)$ is harmonic with respect to the Bergman metric, or simply Bergman-harmonic in Ω if $\Delta_g u = 0$. An important property of the space of Bergman-harmonic functions is its invariance under compositions by automorphisms (i.e., biholomorphic self-mappings) of Ω. Indeed, the Laplace-Beltrami operator Δ_g satisfies the following invariant condition for every $u \in C^2(\Omega)$ and automorphism ψ of Ω:

$$\Delta_g(u \circ \psi) = (\Delta_g u) \circ \psi. \tag{1.1}$$

In general the space of Bergman-harmonic functions properly contains the space of pluriharmonic functions.

A fundamental question on this subject asks what can be the boundary value of Bergman-harmonic functions on a bounded pseudconvex domain Ω and what boundary regularity can be expected. The

*partially supported by National Science Foundation grant DMS-1800549.

problem is subtle since the operator Δ_g is not uniformly elliptic in general and lots of techniques from PDE cannot be applied. It is better-understood in the unit ball case and is widely open in general. When $\Omega = \mathbb{B}^n$, for any $\phi \in C(\partial\mathbb{B}^n)$, the following Poisson integral gives the unique Bergman-harmonic function in $\mathbb{B}^n$ with boundary value ϕ:

$$u(z) = \int_{\partial\mathbb{B}^n} \frac{(1-||z||^2)^n}{|1-\langle z, w\rangle|^{2n}} \phi(w) d\sigma(w).$$

In his seminal work [G1-2], Graham proved that when $\phi \in C^\infty(\partial\mathbb{B}^n)$, the function u given by the above integral satisfies an asymptotic expansion in $\mathbb{B}^n$:

$$u(z) = G(z) + H(z)(1-||z||^2)^n \log(1-||z||^2).$$

Here $G, H \in C^\infty(\overline{\mathbb{B}^n})$. In general, when $n \geq 2, H \not\equiv 0$ on $\partial\mathbb{B}^n$ and consequently $u(z)$ does not have C^n-smooth extension to the boundary. Moreover, corresponding to the case $H \equiv 0$ on $\partial\mathbb{B}^n$, Graham [G1] established the following striking result.

Theorem 1. *(Graham [G1]) If $u \in C^n(\overline{\mathbb{B}^n})$ is Bergman-harmonic ($\Delta_g u = 0$) in $\mathbb{B}^n$, then u is pluriharmonic in $\mathbb{B}^n$.*

Here and in the remaining of the paper, we say $u \in C^m(K)$ for a closed set K if u is C^m in some open set containing K. Since the work of Graham, many authors contributed to understanding the pluriharmonicity of harmonic functions with respect to various interesting metrics and related topics. Here we mention the results of Graham [G1-2], Graham-Lee [GL], Ahern-Bruna-Cascante [ABC], Li-Simon [LS], Li-Wei [LW], Li-Ni [LN] and references therein.

In the remaining part of the paper, we will concentrate on the case when Ω is a classical domain. The study of Bergman-harmonic functions on classical domains dates back to the work of Hua [H] and Lu [Lu]. Let D be a classical domain and write $S(D)$ for the Shilov boundary of D. Hua [H] proved that, for $\phi \in C(S(D))$, the following Poisson integral gives a Bergman-harmonic function on D:

$$u(z) = \int_{S(D)} P(z, w)\phi(w) d\sigma(w).$$

Here $P(z, w)$ is the Poisson-Szegö kernel (cf [H], [Lu] for more details). Conversely, any function $u \in C^2(D) \cap C(\overline{D})$ that is Bergman-harmonic in D has a Possion integral representation as above with ϕ its value restricted on the Shilov boundary. This Poisson integral formula is a major tool in the study of Bergman-harmonic functions on classical domains.

A remarkable step toward understanding Bergman-harmonic functions on classical domains was made by a recent work of Chen-Li [CL]. They established Graham type results for most cases of classical domains. To illustrate, we in particular recall their result for type I domains $D^I_{p,q}$ (See Section 2 for definition of the latter).

Theorem 2. *(Chen-Li, [CL]) Let $1 \leq p \leq q$ and $q \geq 2$. If $u \in C^q(\overline{D^I_{p,q}})$ is Bergman-harmonic in $D^I_{p,q}$, then u is pluriharmonic.*

In the first part (Section 3) of this paper, we explore further along this research line and in particular study the boundary pluriharmonicity of Bergman-harmonic functions on classical domains. To explain our result, we first recall the notion of bounded symmetric domains and their boundary structure. A complex manifold X with a Hermitian metric h is said to be a Hermitian symmetric space if, for every point $z \in X$, there exists an involutive holomorphic isometry σ_z of X such that z is an isolated fixed point. An irreducible Hermitian symmetric space of noncompact type can be, by the Harish-Chandra embedding (See [Wo]), realized as a bounded domain in some complex Euclidean space. Such domains are convex, circular and sometimes called bounded symmetric domains. Irreducible bounded symmetric domains can be classified into Cartan's four types of classical domains and two exceptional domains (See [M]).

The rank r of a bounded symmetric domain D, can be defined as the dimension of the maximal polydisc that can be totally geodesically embedded into D. We next recall the boundary fine structure of an irreducible bounded symmetric domain D ([Wo]). By Borel embedding (See [M], [Wo]), D can be canonically embedded into its dual Hermitian symmetric manifold X of the compact type. Under the embedding, every automorphism $g \in \mathrm{Aut}(D)$ extends to an automorphism of X and D becomes an open orbit under the action of $\mathrm{Aut}(D)$ on X. Moreover, denoting the rank of D by r, the topological boundary ∂D of D decomposes into exactly r orbits under the action of the identity component G of $\mathrm{Aut}(D)$: $\partial D = \cup_{i=1}^r E_i$, where E_k lies in the closure of E_l if $k > l$. Moreover, E_k is the smooth part of the semi-analytic variety $\cup_{j=k}^r E_j$ (See the proof of Lemma 2.2.3 in [MN]). In particular E_1 is the unique open orbit in ∂D, which is indeed the smooth part of ∂D, and E_r is the Shilov boundary. Sometimes we also write $E_0 := D$ so that $\overline{D} = \cup_{i=0}^r E_i$. Note the boundary ∂D of a bounded symmetric domain D is non-smooth and contains complex varieties, unless D is biholomorphic to the unit ball.

We next introduce our results and start with the type I domain case.

Theorem 3. *Let $1 \leq p \leq q$. Fix $0 \leq k \leq p-1$ and set $l = \max\{2, q-k\}$. If $u \in C^l(\overline{D^I_{p,q}})$ is Bergman-harmonic in $D^I_{p,q}$, then u is pluriharmonic on every germ of complex manifold in E_k.*

Remark 1.1.
- When $p = 1$, Theorem 3 is reduced to Graham's theorem (Theorem 1) and when $k = 0$, it is reduced to Chen-Li's Theorem (Theorem 2).
- Theorem 3 is not meaningful if $k = p$ as the Shilov boundary E_p contains only trivial complex varieties.

In particular Theorem 3 implies the following result when $k = 1$.

Corollary 1.1. *Let $1 \leq p \leq q$ and $q \geq 3$. If $u \in C^{q-1}(\overline{D^I_{p,q}})$ is Bergman-harmonic in $D^I_{p,q}$, then u is pluriharmonic on every germ of complex manifold in the boundary of $D^I_{p,q}$.*

We also establish analogous results for other types of classical domains with different boundary regularity assumptions. We leave the detailed statements for other types of domains to Section 3. We only mentione here the following particular result.

Theorem 4. *Let D be a classical domain of type II, III or IV in $\mathbb{C}^N$ with $r = \operatorname{rank}(D) \geq 2$. Assume m is even if D is of type II. Let V be an open set in $\mathbb{C}^N$ containing $\overline{D} - E_r$, where E_r is the Shilov boundary of D. Assume $u \in C^2(V)$ is Bergman-harmonic in D. Then u is harmonic on every germ of complex curve in E_{r-1}. In the type IV case, this means u is harmonic on every germ of complex curve in the boundary ∂D.*

Remark 1.2. Since we do not assume $u \in C(\overline{D})$, we cannot use the Poisson integral formula for Bergman-harmonic functions in the proof.

In the second part of the paper (Section 4), we establish a new characterization result for pluriharmonicity by using the geometric structure of variety of minimal rational tangents of bounded symmetric domains (See Section 4 for the definition).

Theorem 5. *A C^2-smooth function h on an irreducible bounded symmetric domain D is pluriharmonic if and only if h is harmonic on every minimal disk of D.*

We expect this result to be useful in the future study of Bergman-harmonic functions on bounded symmetric domains. We also include some other possible applications in Section 4.

Finally, using ideas introduced in Section 3, we provide a characterization of Bergman-harmonic functions on type I domains in Section 5.

Acknowledgement: The author thanks Sui-Chung Ng for helpful discussions on varieties of minimal rational tangents.

2. Classical domains and the Hua operators

This section includes some preliminaries on classical domain. Recall that classical domains are classified into Cartan's four types of domains, which are defined as follows respectively. Note in this paper we follow the convention in [M], [Wo] and [PS] by which the type II domains correspond to the space of skew-symmetric matrices, while the type III domains correspond to the space of symmetric matrices. These two notions are, however, switched in some other literature, such as [Lu] and [CL].

- Assume $p \leq q$ and write $\mathbb{C}^{p\times q}$ for the space of $p \times q$ matrices with entries of complex numbers. Denote by I_p the $p \times p$ identity matrix. We write a $p \times p$ matrix $A > 0$ if A is strictly positive definite. The classical domain of type I is defined by:
$$D^I_{p,q} = \{Z \in \mathbb{C}^{p\times q} : I_p - Z\overline{Z}^t > 0\}.$$
The boundary of $D^I_{p,q}$ is given by
$$\partial D^I_{p,q} = \{Z \in \mathbb{C}^{p\times q} : I_p - Z\overline{Z}^t \geq 0;\ \det(I_p - Z\overline{Z}^t) = 0\}.$$
Note $D^I_{p,q}$ is of rank p and the boundary $\partial D^I_{p,q}$ decomposes into p orbits under the action of the identity component G_0 of $\operatorname{Aut}(D^I_{p,q}) : \partial D^I_{p,q} = \cup_{i=1}^p E_i$. Here E_k lies in the closure of E_l when $k > l; E_1$ is the smooth part of $\partial D^I_{p,q}$, and E_p is the Shilov boundary. More explicitly in this type I case,
$$E_k = \{Z \in \partial D^I_{p,q} :\ \text{the corank of } I_p - Z\overline{Z}^t \text{ equals } k\}, \quad 1 \leq k \leq p.$$

- Denote by $\mathbb{C}_{II}^{\frac{m(m-1)}{2}} = \{Z \in \mathbb{C}^{m\times m} : Z = -Z^t\}$ the set of all skew-symmetric square matrices of size $m \times m$. Recall the classical domain of type II is defined by

$$D_m^{II} = \{Z \in \mathbb{C}_{II}^{\frac{m(m-1)}{2}} : I_m - Z\overline{Z}^t > 0\}.$$

The boundary of D_m^{III} is given by

$$\partial D_m^{II} = \{Z \in \mathbb{C}_{II}^{\frac{m(m-1)}{2}} : I_m - Z\overline{Z}^t \geq 0;\ \det(I_m - Z\overline{Z}^t) = 0\}.$$

Note the rank of D_m^{II} equals $r = \lfloor \frac{1}{2}m \rfloor$. Here $\lfloor \cdot \rfloor$ denotes the floor function, i.e., $2r = m$ if m is even and $2r + 1 = m$ if m is odd. The boundary ∂D_m^{II} decomposes into r orbits under the action of the identity component G_0 of $\mathrm{Aut}(D_m^{II}) : \partial D_m^{II} = \cup_{i=1}^r E_i$. Here E_k lies in the closure of E_l if $k > l$. More explicitly in this type II case (See [Wo], [X] for more details),

$$E_k = \{Z \in \partial D_m^{II} :\ \text{the corank of } I_m - Z\overline{Z}^t \text{ equals } 2k\}, \quad 1 \leq k \leq r.$$

- Denote by $\mathbb{C}_{III}^{\frac{m(m+1)}{2}} = \{Z \in \mathbb{C}^{m\times m} : Z = Z^t\}$ the set of all symmetric square matrices of size $m \times m$. Recall the classical domain of type III is defined by

$$D_m^{III} = \{Z \in \mathbb{C}_{III}^{\frac{m(m+1)}{2}} : I_m - Z\overline{Z}^t > 0\}.$$

The boundary of D_m^{III} is given by

$$\partial D_m^{III} = \{Z \in \mathbb{C}_{III}^{\frac{m(m+1)}{2}} : I_m - Z\overline{Z}^t \geq 0;\ \det(I_m - Z\overline{Z}^t) = 0\}.$$

Note D_m^{III} is of rank m and the boundary ∂D_m^{III} decomposes into m orbits under the action of the identity component G_0 of $\mathrm{Aut}(D_m^{III}) : \partial D_m^{III} = \cup_{i=1}^m E_i$. Here E_k lies in the closure of E_l when $k > l$. More explicitly in this type III case,

$$E_k = \{Z \in \partial D_m^{III} :\ \text{the corank of } I_m - Z\overline{Z}^t \text{ equals } k\}, \quad 1 \leq k \leq m.$$

- Recall the type IV classical domain $D_m^{IV} (m \geq 2)$, often called the Lie ball, is defined by

$$D_m^{IV} = \{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : Z\overline{Z}^t < 1,\ 1 - 2Z\overline{Z}^t + |ZZ^t|^2 > 0\}.$$

When $m = 2, D_2^{IV}$ is biholomorphic to the bidisc. The boundary of D_m^{IV} is given by

$$\partial D_m^{IV} = \{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : Z\overline{Z}^t \leq 1, 1 - 2Z\overline{Z}^t + |ZZ^t|^2 = 0\}.$$

Since the type IV domain D_m^{IV} is always of rank two, its boundary is stratified into two orbits: $\partial D_m^{IV} = E_1 \cup E_2$, where E_1 is the smooth boundary and E_2 is the Shilov boundary of D_m^{IV}. Here

$$E_1 = \{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : Z\overline{Z}^t < 1, 1 - 2Z\overline{Z}^t + |ZZ^t|^2 = 0\};$$

$$E_2 = \{Z \in \mathbb{C}^m : Z\overline{Z}^t = 1, 1 - 2Z\overline{Z}^t + |ZZ^t|^2 = 0\} = \{Z \in \mathbb{C}^m : ||Z||^2 = |ZZ^t| = 1\}. \tag{2.1}$$

Write $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 for the Hua operators associated to the four types of classical domains, respectively (See [H], [Lu]). Recall they equal to the Laplace-Beltrami operator with respect to the Bergman metrics (up to a constant multiplication). We include here the explicit expressions for the Hua operators which can be found in [H], [Lu].

- Type I case: Let $Z = (z_{ij})_{1\le i\le p, 1\le j\le q} \in \mathbb{C}^{p\times q}$. Write

$$V(Z) = I_p - Z\overline{Z}^t; \qquad V_{jk} = [V(Z)]_{jk} = \delta_{jk} - \sum_{l=1}^{q} z_{jl}\overline{z}_{kl},\ 1 \le j,k \le p. \tag{2.2}$$

  Then the Hua operator is given by

$$\Delta_1 = \sum_{j,k=1}^{p} V_{jk}\Delta_1^{jk}, \qquad \Delta_1^{jk} := \sum_{\alpha,\beta=1}^{q}\left(\delta_{\alpha\beta} - \sum_{l=1}^{p} z_{l\alpha}\overline{z}_{l\beta}\right)\frac{\partial^2}{\partial z_{j\alpha}\partial \overline{z}_{k\beta}}; \tag{2.3}$$

- Type II case: Let $Z = (z_{ij})_{1\le i,j\le m} \in \mathbb{C}_{II}^{\frac{m(m-1)}{2}}$ and V_{jk} be as in (2.2). Then the Hua operator is given by

$$\Delta_2 = \frac{1}{4}\sum_{j,k=1}^{m} V_{jk}\Delta_2^{jk}; \qquad \Delta_2^{jk} := \sum_{j,k=1}^{m} V_{\alpha\beta}(1-\delta_{j\alpha})(1-\delta_{k\beta})\frac{\partial^2}{\partial z_{j\alpha}\partial \overline{z}_{k\beta}}. \tag{2.4}$$

- Type III case: Let $Z = (z_{ij})_{1\le i,j\le m} \in \mathbb{C}_{III}^{\frac{m(m+1)}{2}}$ and V_{jk} be as in (2.2). Then the Hua operator is given by

$$\Delta_3 = \frac{1}{4}\sum_{j,k=1}^{m} V_{jk}\Delta_2^{jk}, \qquad \Delta_3^{jk} := \sum_{\alpha,\beta=1}^{m} \frac{V_{\alpha\beta}}{(1-\delta_{j\alpha}/2)(1-\delta_{k\beta}/2)}\frac{\partial^2}{\partial z_{j\alpha}\partial \overline{z}_{k\beta}}. \tag{2.5}$$

- Type IV case: Let $Z = (z_1, \cdots, z_n) \in \mathbb{C}^n$. Write

$$s(Z) = \sum_{j=1}^{n} z_j^2, \qquad r(Z) = 1 - 2\|Z\|^2 + |s(Z)|^2 \tag{2.6}$$

  Then the Hua operator is given by

$$\Delta_4 = \sum_{j,k=1}^{n} \left[r(Z)(\delta_{jk} - 2z_j\overline{z}_k) + 2(\overline{z}_j - s(\overline{Z})z_j)(z_k - s(Z)\overline{z}_k)\right]\frac{\partial^2}{\partial z_j\partial \overline{z}_k}. \tag{2.7}$$

3. Boundary Pluriharmonicity

In this section, we establish boundary pluriharmonicity results for each type of classical domains. The proof relies on the biholomorphic invariance of Bergman-harmonic functions and fundamentally uses the structure of analytic sets in the boundary of a classical domain. We start with the notion of boundary components of a domain Ω in $\mathbb{C}^n$(cf. [PS]).

Definition 3.1. Let $\Omega \subset \mathbb{C}^n$ be a domain, and $\partial\Omega$ its topological boundary. An analytic set $X \subset \partial\Omega$ is called a boundary component of Ω if any analytic curve that is entirely contained in $\partial\Omega$ and intersects X must lie entirely in X.

We say a boundary component X is regular if it is smooth as an analytic set. We recall the following result for classical domains from [PS].

Proposition 3.1. *(Theorem 5, page 85, [PS]) Let D be a classical domain in $\mathbb{C}^N$. Then the following hold:*

(1) Every point in ∂D is contained in some boundary component.
(2) Every boundary component of D is regular and furthermore biholomorphically equivalent to some classical domain in a complex space of lower dimension.

We remark that there are only trivial analytic varieties in the Shilov boundary of D, and in the above theorem, each point in the Shilov boundary is counted as a boundary component. We will call them trivial boundary components.

We introduce the following defintion to study the boundary behavior of Bergman-harmonic functions on classical domains.

Definition 3.2. Let D be a classical domain in $\mathbb{C}^N$ and $D' \subset \mathbb{C}^{N'}$ a classical domain of lower dimension than D.

(1) We say $h : D' \to \partial D$ gives a boundary component of D if the image $h(D')$ is some boundary component of D and h is a biholomorphism from D' to $h(D')$.
(2) Let $u \in C^2(U)$ for some open subset U of $\mathbb{C}^N$ containing $h(D')$. We say u is Bergman-harmonic on the boundary component $h(D')$ if the composition $u \circ h$ is Bergman-harmonic in D'.

Remark 3.1. We make a couple of important remarks on this definition.

- Part (1) makes sense due to Proposition 3.1.
- The notion in part (2) is independent of the choice of h and D'. Indeed, suppose $\hat{h} : \hat{D} \to \partial D$ gives the same boundary component, i.e., $\hat{h}(\hat{D}) = h(D')$. Then we must have $\hat{h} = h \circ \phi$ for some biholomorphism ϕ from $\hat{D}$ to D'. It then follows from the biholomorphic invariance of Bergman-harmonic functions that $u \circ \hat{h}$ is Bergman-harmonic in $\hat{D}$ if and only if $u \circ h$ is so in D'.

The following result plays a key role in establishing the boundary pluriharmonicity.

Theorem 6. *Let $D \subset \mathbb{C}^N$ be a classical domain and V an open subset of $\mathbb{C}^N$ containing the set $\overline{D} - E_r$, where E_r is the Shilov boundary of D. Assume $u \in C^2(V)$ is Bergman-harmonic in D. Then u is Bergman-harmonic on every boundary component of D. That is, for every $h : D' \to D$ which gives a boundary component of D, it holds that $u \circ h$ is Bergman-harmonic in D'.*

Remark 3.2. (1) Here we do not assume C^2-smoothness of u across the Shilov boundary E_r since there are only trivial boundary components contained in E_r.

(2) If we assume additionally that $u \in C(\overline{D})$, then one can use the Poisson integral formula for Bergman-harmonic functions and apply results in [Lu] to prove Theorem 6. Our proof will, however, avoid the Poisson integral formula as we do not assume the continuity of u on $\overline{D}$.

Due to the distinct boundary structures of different types of classical domains, we will prove Theorem 6 case by case for the four types of classical domains in subsections 3.1-3.4.

3.1. **Boundary pluriharmonicity on type I domains.** In this subsection, we prove Theorem 6 for type I case and establish Theorem 3. Assume $D = D^I_{p,q}$ for some $1 \leq p \leq q$. Recall the following facts from [PS]. Denote by I_k the $k \times k$ identity matrix.

Lemma 3.1. *(Theorem 1, page 93, [PS])*

(1) Let $1 \leq k \leq p$ and X_k be the set of points in $\partial D^I_{p,q}$ of form:

$$\begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0} & W \end{pmatrix}, \qquad \text{where } W \in D^I_{p-k,q-k}. \tag{3.1}$$

Then X_k is a boundary component of $D^I_{p,q}$. Clearly $X_k \simeq D^I_{p-k,q-k}$.

(2) Every boundary component X of $D^I_{p,q}$ can be transformed to X_k for some $1 \leq k \leq p$ by some automorphism of $D^I_{p,q}$.

We will call such boundary component X in (2) a boundary $k-$component. Note a boundary $k-$component lies entirely in $E_k \subset \partial D^I_{p,q}$. We call the boundary component given by $h_1 : D^I_{p-1,q-1} \to \partial D^I_{p,q}$ a standard $1-$component where

$$h_1(W) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & W \end{pmatrix} \in X_1, \qquad W \in D^I_{p-1,q-1}. \tag{3.2}$$

Note if u extends C^2-smoothly across the smooth part E_1 of $\partial D^I_{p,q}$, then $u \circ h_1$ is C^2 on $D^I_{p-1,q-1}$. We first show

Lemma 3.2. *Let V an open subset in $\mathbb{C}^{p\times q}$ containing $D^I_{p,q}$ and E_1, where E_1 is the smooth boundary of $D^I_{p,q}$. Assume $u \in C^2(V)$ is Bergman-harmonic in $D^I_{p,q}$. Then $u \circ h_1$ is Bergman-harmonic on $D^I_{p-1,q-1}$.*

Proof. Write $W = (w_{st})_{1\leq s\leq p-1, 1\leq t\leq q-1}$ for the coordinates in $D^I_{p-1,q-1} \subset \mathbb{C}^{(p-1)\times(q-1)}$. Recall by equation (2.3) the Hua operator on $D^I_{p-1,q-1}$ is given by

$$\widetilde{\Delta}_1 = \sum_{s,t=1}^{p-1} \widetilde{V}_{st} \widetilde{\Delta}_1^{st}. \tag{3.3}$$

Here $\widetilde{V}_{st} = \widetilde{V}_{st}(W) = \delta_{st} - \sum_{r=1}^{q-1} w_{sr}\overline{w}_{tr}$; $\quad \widetilde{\Delta}_1^{st} = \sum_{a,b=1}^{q-1}(\delta_{ab} - \sum_{r=1}^{p-1} w_{ra}\overline{w}_{rb})\frac{\partial^2}{\partial w_{sa}\partial \overline{w}_{tb}}$.

We need to show $\widetilde{\Delta}_1(u \circ h_1) = 0$ on $D^I_{p-1,q-1}$. For that we first note by assumption that $\Delta_1 u = 0$ where Δ_1 is given by (2.3). Since $u \in C^2(V)$, and the coefficients in the differential operater Δ_1 is smooth in $\mathbb{C}^{p\times q}$, we conclude $\Delta_1 u = 0$ in V. Next we fix $W_0 \in D^I_{p-1,q-1}$ and $Z_0 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & W_0 \end{pmatrix} \in$

$X_1 \subset E_1 \subset V$. We evaluate $\Delta_1 u$ at $Z = (z_{ij})_{1\le i\le p, 1\le j\le q} = Z_0$. It must equal zero by the preceding argument. On the other hand, using (2.2)-(2.3), we have

$$\begin{gathered} V_{11}\big|_{Z_0} = 1 - (\sum_{l=1}^{q} z_{1l}\overline{z}_{1l})\big|_{Z_0} = 0. \\ V_{1k}\big|_{Z_0} = -(\sum_{l=1}^{q} z_{1l}\overline{z}_{kl})\big|_{Z_0} = 0 \text{ if } k > 1; \quad V_{j1}\big|_{Z_0} = -(\sum_{l=1}^{q} z_{jl}\overline{z}_{1l})\big|_{Z_0} = 0 \text{ if } j > 1. \end{gathered} \tag{3.4}$$

When $j, k > 1$,

$$\begin{aligned} V_{jk}\big|_{Z_0} &= \delta_{jk} - (\sum_{l=1}^{q} z_{jl}\overline{z}_{kl})\big|_{Z_0} = \delta_{(j-1)(k-1)} - (\sum_{l=2}^{q} z_{jl}\overline{z}_{kl})\big|_{Z_0} \\ &= \delta_{(j-1)(k-1)} - \left(\sum_{r=1}^{q-1} w_{(j-1)r}\overline{w}_{r(k-1)}\right)\big|_{W_0} = \widetilde{V}_{(j-1)(k-1)}\big|_{W_0}. \end{aligned} \tag{3.5}$$

Next we note

$$\Delta_1^{jk} u\big|_{Z_0} = \sum_{\alpha,\beta=1}^{q} (\delta_{\alpha\beta} - \sum_{l=1}^{p} z_{l\alpha}\overline{z}_{l\beta}) \frac{\partial^2 u}{\partial z_{j\alpha}\overline{z}_{k\beta}}\big|_{Z_0}, \tag{3.6}$$

and compute the term in the parenthesis in (3.6):

- When $\alpha = 1, \beta = 1$, $\delta_{11} - (\sum_{l=1}^{p} z_{l1}\overline{z}_{l1})\big|_{Z_0} = 0$;
- When $\alpha = 1, \beta > 1$, $\delta_{1\beta} - (\sum_{l=1}^{p} z_{l1}\overline{z}_{l\beta})\big|_{Z_0} = 0$;
- When $\alpha > 1, \beta = 1$, $\delta_{\alpha 1} - (\sum_{l=1}^{p} z_{l\alpha}\overline{z}_{l1})\big|_{Z_0} = 0$;
- When $\alpha > 1, \beta > 1$, we have
  $\delta_{\alpha\beta} - (\sum_{l=1}^{p} z_{l\alpha}\overline{z}_{l\beta})\big|_{Z_0} = \delta_{\alpha\beta} - (\sum_{l=2}^{p} z_{l\alpha}\overline{z}_{l\beta})\big|_{Z_0} = \delta_{(\alpha-1)(\beta-1)} - (\sum_{r=1}^{p-1} w_{r(\alpha-1)}\overline{w}_{r(\beta-1)})\big|_{W_0}$.

Moreover, we have by chain rule,

$$\frac{\partial^2 u}{\partial z_{j\alpha}\partial\overline{z}_{k\beta}}\big|_{Z_0} = \frac{\partial^2 (u\circ h_1)}{\partial w_{(j-1)(\alpha-1)}\partial\overline{w}_{(k-1)(\beta-1)}}\big|_{W_0} \text{ if } j > 1, k > 1, \alpha > 1, \beta > 1.$$

Substituting these into (3.6), we obtain when $j > 1, k > 1$,

$$\begin{aligned} \Delta_1^{jk} u\big|_{Z_0} &= \sum_{\alpha,\beta=2}^{q} (\delta_{\alpha\beta} - \sum_{l=1}^{p} z_{l\alpha}\overline{z}_{l\beta}) \frac{\partial^2 u}{\partial z_{j\alpha}\partial\overline{z}_{k\beta}}\big|_{Z_0} \\ &= \sum_{a,b=1}^{q-1} (\delta_{ab} - \sum_{r=1}^{p-1} w_{ra}\overline{w}_{rb}) \frac{\partial^2 (u\circ h_1)}{\partial w_{(j-1)a}\partial\overline{w}_{(k-1)b}}\big|_{W_0} \\ &= \widetilde{\Delta}_1^{(j-1)(k-1)} (u\circ h_1)\big|_{W_0}. \end{aligned} \tag{3.7}$$

Consequently, by (3.4), (3.5) and (3.7),

$$\Delta_1 u\big|_{Z_0} = \sum_{j,k=2}^{p} V_{jk}\Delta_1^{jk} u\big|_{Z_0} = \sum_{s,t=1}^{p-1} \widetilde{V}_{st}\widetilde{\Delta}_1^{st}(u\circ h_1)\big|_{W_0} = 0.$$

This implies $\widetilde{\Delta}_1(u\circ h_1)=0$ at $W=W_0$ by (3.3) Since W_0 is arbitrary in $D^I_{p-1,q-1}$, we conclude $u\circ h_1$ is Bergman-harmonic in $D^I_{p-1,q-1}$.

Proof of Theorem 6 for the type I case: Let $D=D^I_{p,q}$ with $1\le p\le q$ and V be as in Theorem 6. Let $u\in C^2(V)$ be Bergman-harmonic in $D^I_{p,q}$. By Lemma 3.2, $u\circ h_1$ is Bergman-harmonic on $D^I_{p-1,q-1}$, where h_1 gives the standard $1-$component as in (3.2). Let $h: D^I_{p-1,q-1}\to\partial D^I_{p,q}$ be another boundary $1-$component. By Proposition 3.1, $h=\psi\circ h_1\circ\phi$ for some $\phi\in \mathrm{Aut}(D^I_{p-1,q-1})$ and $\psi\in\mathrm{Aut}(D^I_{p,q})$. Since ψ extends holomorphically to a neighborhood of $\overline{D^I_{p,q}}$, there exists an open set $\hat{V}$ in $\mathbb{C}^{p\times q}$ containing $D^I_{p,q}$ and E_1 such that $u\circ\psi\in C^2(\hat{V})$. Moreover, $u\circ\psi$ is Bergman-harmonic in $D^I_{p,q}$ by the biholomorphic invariance of Bergman-harmonic functions. We conclude by Lemma 3.2 that $u\circ\psi\circ h_1$ is Bergman-harmonic and consequently $u\circ h=u\circ\psi\circ h_1\circ\phi$ is Bergman-harmonic in $D^I_{p-1,q-1}$. We have thus proved u is Bergman-harmonic on every boundary $1-$component of $D^I_{p,q}$.

For $2\le k\le p-1$, we note every boundary $k-$component Y_k of $D^I_{p,q}$ is a boundary $1-$component of some boundary $(k-1)-$component Y_{k-1} of $D^I_{p,q}$. Moreover, the smooth boundary of Y_{k-1} lies in $E_k\subset\partial D^I_{p,q}$. Thus it follows from the boundary regularity assumption and an inductive application of Lemma 3.2 that u is Bergman-harmonic on every boundary component of $D^I_{p,q}$. One can prove the general case by direct computation as well by using the explicit expression (3.1) and biholomorphic invariance of Bergman-harmonic functions.

□

We next prove the boundary pluriharmonicity of Bergman-harmonic functions on type I domains, i.e., Theorem 3. We first make the following remark.

Remark 3.3. Let $h_k: D^I_{p-k,q-k}\subset\mathbb{C}^{(p-k)\times(q-k)}\to D^I_{p,q}$ be as in (3.1) that gives the standard $k-$component. Then clearly h_k extends to be a polynomial map in $\mathbb{C}^{(p-k)\times(q-k)}$. More generally, assume $h: D^I_{p-k,q-k}\subset\mathbb{C}^{(p-k)\times(q-k)}\to D^I_{p,q}$ gives any boundary $k-$component. By Proposition 3.1, $h=\psi\circ h_k\circ\phi$ for some $\phi\in\mathrm{Aut}(D^I_{p-k,q-k})$ and $\psi\in\mathrm{Aut}(D^I_{p,q})$. Since ϕ and ψ extend holomorphically to a neighborhood of $\overline{D^I_{p-k,q-k}}$ and $\overline{D^I_{p,q}}$, respectively. As a consequence, h extends holomorphically to a neighborhood of $\overline{D^I_{p-k,q-k}}$.

Proof of Theorem 3: Fix $1\le k\le p-1$. To prove Theorem 3, we first note every analytic curve in E_k is contained in some boundary component of $D^I_{p,q}$ in E_k and, in light of Proposition 3.1, these boundary components are precisely boundary $k-$components of $D^I_{p,q}$. It thus suffices to show u is pluriharmonic in every boundary $k-$component. Assume $h: D^I_{p-k,q-k}\to\partial D^I_{p,q}$ gives a boundary $k-$component of $D^I_{p,q}$. Theorem 6 implies $u\circ h$ is Bergman-harmonic in $D^I_{p-k,q-k}$. Moreover, by assumption, $u\in C^l(\overline{D^I_{p,q}})$, with $l=\max\{2,q-k\}$. If $q-k<2$, then $p=q=k+1$ and $D^I_{p-k,q-k}$ is the unit disk Δ and the conclusion follows easily. Now assume $q-k\ge 2$. Consequently, $u\circ h$ extends $(q-k)-$smoothly across the boundary of $D^I_{p-k,q-k}$ by Remark 3.3. We finally apply Theorem 1.2 (i) in [CL] to conclude $u\circ h$ is pluriharmonic in $D^I_{p-k,q-k}$. This establishes Theorem 3. ■

3.2. **Boundary pluriharmonicity on type II domains.** In this subsection, we prove Theorem 6 for type II domains and establish boundary pluriharmonicity for Bergman-harmonic functions on type II domains $D = D_m^{II}$ for $m \geq 2$. Recall the following result from [PS].

Lemma 3.3. *(Theorem 1, page 115, [PS])*

(1) *Let the integer $1 \leq k \leq \frac{m}{2}$ and X_k be the set of points ∂D_m^{II} of the form:*

$$\begin{pmatrix} J & & & & \\ & J & & & \\ & & \cdots & & \\ & & & J & \\ & & & & W \end{pmatrix}$$

where there are k copies of $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $W \in D_{m-2k}^{II}$, and all other omitted elements are zero. Then X_k is a boundary component of D_m^{II}.

(2) *Any boundary component X of D_m^{II} can be transformed to X_k for some integer $1 \leq k \leq \frac{m}{2}$ by some automorphism of D_m^{II}.*

We will call such X in (2) a boundary $k-$component of D_m^{II}. Note every boundary $k-$component lies in E_k. In particular, we say $h_1 : D_{m-2}^{II} \to \partial D_m^{II}$ gives the standard boundary $1-$component, where

$$h_1(W) = \begin{pmatrix} J & \\ & W \end{pmatrix}, \quad W \in D_{m-2}^{II} \subset \mathbb{C}_{II}^{\frac{(m-2)(m-3)}{2}}.$$

As before, note $u \circ h_1 \in C^2(D_{m-2}^{II})$ if u extends C^2-smoothly across the smooth boundary E_1 of D_m^{II}, and prove

Lemma 3.4. *Let V be an open set in $\mathbb{C}_{II}^{\frac{m(m-1)}{2}}$ containing D_m^{II} and E_1. Assume $u \in C^2(V)$ is Bergman-harmonic in D_m^{II}. Then $u \circ h_1$ is Bergman-harmonic in D_{m-2}^{II}.*

Proof. Write $W = (w_{ij})_{1 \leq i,j \leq m-2}$ with $w_{ij} = -w_{ji}$ for the coordinates in $\mathbb{C}_{II}^{\frac{(m-2)(m-3)}{2}}$. Recall by equation (2.4), the Hua operator on D_{m-2}^{II} is given by

$$\widetilde{\Delta}_2 = \frac{1}{4} \sum_{s,t=1}^{m-2} \widetilde{V}_{st} \widetilde{\Delta}_2^{st}, \tag{3.8}$$

where $\widetilde{V}_{st} = \delta_{st} - \sum_{r=1}^{m-2} w_{sr} \overline{w}_{tl}$; $\quad \widetilde{\Delta}_2^{st} = \sum_{a,b=1}^{m-2} \widetilde{V}_{ab}(1 - \delta_{sa})(1 - \delta_{tb}) \frac{\partial^2}{\partial w_{sa} \partial \overline{w}_{tb}}$.

We will prove $\widetilde{\Delta}_2(u \circ h_1) = 0$ on D_{m-2}^{II}. Let Δ_2 be the Hua operator on the type II domain D_m^{II} as in (2.4). As in the proof of Lemma 3.2, we have $\Delta_2 u = 0$ on E_1 by the Bergman-harmonicity and boundary regularity of u. Consequently, if we fix $W_0 \in D_{m-2}^{II}$, and $Z_0 = \begin{pmatrix} J & 0 \\ 0 & W_0 \end{pmatrix} \in X_1 \subset E_1$, then $\Delta_2 u = 0$ at $Z = Z_0$. We next compute $\Delta_2 u$ at $Z = Z_0$ by using (2.4).

Claim: $V_{jk}|_{Z_0} = 0$ if $j \le 2$ or $k \le 2$. If $j > 2$ and $k > 2$, then $V_{jk}\big|_{Z_0} = \widetilde{V}_{(j-2)(k-2)}\big|_{W_0}$.

Proof of Claim: If $j = k \le 2$, we have

$$V_{11}\big|_{Z_0} = 1 - (\sum_{l=1}^{m} z_{1l}\overline{z}_{1l})\big|_{Z_0} = 0; \quad V_{22}\big|_{Z_0} = 1 - (\sum_{l=1}^{m} z_{2l}\overline{z}_{2l})\big|_{Z_0} = 0.$$

If $j \ne k$ and one of them is at most 2, we have

$$V_{jk}\big|_{Z_0} = -(\sum_{l=1}^{m} z_{jl}\overline{z}_{kl})\big|_{Z_0} = 0.$$

If $j > 2$ and $k > 2$, then

$$\begin{aligned} V_{jk}\big|_{Z_0} &= \delta_{jk} - \sum_{l=1}^{m} (z_{jl}\overline{z}_{kl})\big|_{Z_0} = \delta_{jk} - \sum_{l=3}^{m} (z_{jl}\overline{z}_{kl})\big|_{Z_0} \\ &= \delta_{(j-2)(k-2)} - \sum_{l=3}^{m} (w_{(j-2)(l-2)}\overline{w}_{(k-2)(l-2)})\big|_{W_0} \\ &= \widetilde{V}_{(j-2)(k-2)}\big|_{W_0}. \end{aligned} \tag{3.9}$$

This proves the claim. ∎

Note also by the chain rule, if $j > 2, k > 2, \alpha > 2, \beta > 2$,

$$\frac{\partial^2 u}{\partial z_{j\alpha}\partial \overline{z}_{k\beta}}\big|_{Z_0} = \frac{\partial^2 (u \circ h_1)}{\partial w_{(j-2)(\alpha-2)}\partial \overline{w}_{(k-2)(\beta-2)}}\big|_{W_0}.$$

Consequently, if $j > 2$ and $k > 2$,

$$\begin{aligned} \Delta_2^{jk} u\big|_{Z_0} &= \sum_{\alpha,\beta=1}^{m} V_{\alpha\beta}(1-\delta_{j\alpha})(1-\delta_{k\beta})\frac{\partial^2 u}{\partial z_{j\alpha}\partial \overline{z}_{k\beta}}\big|_{Z_0} = \sum_{\alpha,\beta=3}^{m} V_{\alpha\beta}(1-\delta_{j\alpha})(1-\delta_{k\beta})\frac{\partial^2 u}{\partial z_{j\alpha}\partial \overline{z}_{k\beta}}\big|_{Z_0} \\ &= \sum_{\alpha,\beta=3}^{m} \widetilde{V}_{(\alpha-2)(\beta-2)}(1-\delta_{(j-2)(\alpha-2)})(1-\delta_{(k-2)(\beta-2)})\frac{\partial^2 (u \circ h_1)}{\partial w_{(j-2)(\alpha-2)}\partial \overline{w}_{(k-2)(\beta-2)}}\big|_{W_0} \\ &= \widetilde{\Delta}_2^{(j-2)(k-2)}(u \circ h_1)\big|_{W_0}. \end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned} \Delta_2 u\big|_{Z_0} &= \frac{1}{4}\sum_{j,k=1}^{m} V_{jk}\big|_{Z_0}(\Delta_2^{jk})\big|_{Z_0} = \frac{1}{4}\sum_{j,k=3}^{m} V_{jk}\big|_{Z_0}(\Delta_2^{jk})\big|_{Z_0} \\ &= \frac{1}{4}\sum_{s,t=1}^{m-2} \widetilde{V}_{st}\big|_{W_0}\widetilde{\Delta}_2^{st}(u \circ h_1)\big|_{W_0} = 0 \end{aligned} \tag{3.11}$$

This implies $\widetilde{\Delta}_2(u \circ h_1)\big|_{W_0} = 0$ by (3.8). Since W_0 is arbitrary in D^{II}_{m-2}, we have $u \circ h_1$ is Bergman-harmonic in D^{II}_{m-2}. □

Proof of Theorem 6 for the type II case: Theorem 6 in this case can be proved by the same argument as in the type I case and by using Lemma 3.4.

As a consequence of Theorem 6 in this case, we have the following boundary pluriharmonicity for Bergman-harmonic functions on type II domains.

Proposition 3.2. *If $m \geq 6$ is even and $u \in C^{m-3}(\overline{D_m^{II}})$ is Bergman-harmonic in D_m^{II}, then u is pluriharmonic on every germ of complex manifold in ∂D_m^{II}.*

Proof. It suffices to prove that u is pluriharmonic on every boundary component of D_m^{II}. Let $r = \mathrm{rank}(D_m^{II})$. Fix $1 \leq k \leq r-1$. Assume $h : D_{m-2k}^{II} \to \partial D_m^{II}$ gives a boundary $k-$component. Theorem 6 yields $u \circ h$ is Bergman-harmonic on D_{m-2k}^{II}. By assumption, $u \in C^{m-3}(\overline{D_m^{II}})$. Consequently, since h extends holomorphically across the boundary of D_{m-2k}^{II} (This can be seen similarly as in Remark 3.3), we have $u \circ h_1$ extends $(m-3)-$smoothly to a neighborhood of $\overline{D_{m-2k}^{II}}$. We then apply Theorem 1.2 (iv) in [CL] to conclude $u \circ h$ is pluriharmonic in D_{m-2k}^{II}. Since k is arbitrarily chosen, we have proved that u is pluriharmonic on every boundary component, and Proposition 3.2 follows readily. □

Write $r = \mathrm{rank}(D_m^{II})$. Note every germ of nontrivial complex variety in $E_{r-1} \subset \partial D_m^{II}$ must be one-dimensional.

Proposition 3.3. *Let $m \geq 4$ be even and write $r = \mathrm{rank}(D_m^{II})$. Let V be an open set in $\mathbb{C}_{II}^{\frac{m(m-1)}{2}}$ containing $\overline{D_m^{II}} - E_r$. Assume $u \in C^2(V)$ is Bergman-harmonic in D_m^{II}. Then u must be harmonic on every germ of complex curve in E_{r-1}.*

Proof. Note every boundary component contained in E_{r-1} is a boundary $(r-1)-$component of D_m^{II} and it is always of one-dimensional when m is even. That is, every boundary component in E_{r-1} is given by some $h : D_2^{II} \approx \Delta \to \partial D_m^{II}$, where Δ is the unit disk in $\mathbb{C}$(See Lemma 3.3). But Theorem 6 implies $u \circ h$ is Bergman-harmonic on Δ. This is equivalent to harmonicity on Δ. We thus establish Proposition 3.3. □

3.3. **Boundary pluriharmonicity on type III domains.** In this subsection, we prove Theorem 6 for type II domains and establish boundary pluriharmonicity for Bergman-harmonic functions on type III domains $D = D_m^{III}$. Recall the following description on boundary components of D_m^{III} from [PS].

Lemma 3.5. *(Lemma 1, page 125, [PS])*

1. *Let $1 \leq k \leq p$ and X_k be the set of points in $\partial D_{p,q}^{I}$ of form:*
$$\begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0} & W \end{pmatrix}, \qquad \text{where } W \in D_{m-k}^{III}. \tag{3.12}$$
Then X_k is a boundary component of D_m^{III}. Clearly $X_k \simeq D_{m-k}^{III}$.
2. *Every boundary component X of D_m^{I} can be transformed to X_k for some $1 \leq k \leq m$ by some automorphism of D_m^{III}.*

We will call such boundary component X in (2) a boundary $k-$component and call the boundary component given by $h_1 : D^{III}_{m-1} \to \partial D^{III}_m$ a standard $1-$component where

$$h_1(W) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & W \end{pmatrix}, \qquad \text{where } W \in D^{III}_{m-1}.$$

Note every boundary $k-$component lies in E_k.

Lemma 3.6. *Let V be an open set in $\mathbb{C}_{III}^{\frac{m(m+1)}{2}}$ containing D^{III}_m and E_1. Let $u \in C^2(V)$ be Bergman-harmonic. Then $u \circ h_1$ is Bergman-harmonic on D^{III}_{m-1}.*

Proof. Write $W = (w_{ij})_{1\le i,j\le m-1}$ with $w_{ij} = w_{ji}$ for the coordinates in $\mathbb{C}_{III}^{\frac{m(m+1)}{2}}$. Recall by equation (2.5), the Hua operator on D^{III}_{m-1} is given by

$$\widetilde{\Delta}_3 = \frac{1}{4}\sum_{s,t=1}^{m-1} \widetilde{V}_{st}\widetilde{\Delta}_3^{st},$$

where $\widetilde{V}_{st} = \delta_{st} - \sum_{r=1}^{m-1} w_{sr}\overline{w}_{tr}$, and $\widetilde{\Delta}_3^{st} = \sum_{a,b=1}^{m-1} \frac{\widetilde{V}_{ab}}{(1-\frac{\delta_{ab}}{2})(1-\frac{\delta_{tb}}{2})}\frac{\partial^2}{\partial w_{sa}\partial \overline{w}_{tb}}$. We will need to show $\widetilde{\Delta}_3(u\circ h_1) = 0$ on D^{III}_{m-1}. Let Δ_3 be the Hua operator on the type III domain D^{III}_m as in (2.5). As in the proof of Lemma 3.2 and 3.4, we have $\Delta_3 u = 0$ on E_1 by the Bergman-harmonicity and boundary regularity of u. Thus if we fix $W_0 \in D^{III}_{m-1}$, and $Z_0 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & W_0 \end{pmatrix} \in X_1 \subset E_1 \subset \partial D^{III}_m$, then $\Delta_3 u = 0$ at $Z = Z_0$. Next we compute $\Delta_3 u$ at $Z = Z_0$ by using (2.5). Recall $Z = (z_{ij})_{1\le i,j\le m}$. Note

$$V_{11}\big|_{Z_0} = 1 - (\sum_{l=1}^{m} z_{1l}\overline{z}_{1l})\big|_{Z_0} = 1 - 1 = 0.$$

$$V_{1k}\big|_{Z_0} = -(\sum_{l=1}^{m} z_{1l}\overline{z}_{kl})\big|_{Z_0} = 0, \text{ if } k > 1; \qquad V_{j1}\big|_{Z_0} = -(\sum_{l=1}^{m} z_{jl}\overline{z}_{1l})\big|_{Z_0} = 0, \text{ if } j > 1.$$

When $j > 1, k > 1$,

$$V_{jk}\big|_{Z_0} = \delta_{jk} - (\sum_{l=1}^{m} z_{jl}\overline{z}_{kl})\big|_{Z_0} = \delta_{(j-1)(k-1)} - (\sum_{l=2}^{m} w_{(j-1)(l-1)}\overline{w}_{(k-1)(l-1)})\big|_{W_0} = \widetilde{V}_{(j-1)(k-1)}\big|_{W_0}.$$

Note also

$$\frac{\partial^2 u}{\partial z_{j\alpha}\partial \overline{z}_{k\beta}}\big|_{Z_0} = \frac{\partial^2 (u\circ h_1)}{\partial w_{(j-1)(\alpha-1)}\partial \overline{w}_{(k-1)(\beta-1)}}\big|_{W_0}, \text{ if } j>1, k>1, \alpha>1, \beta>1.$$

Consequently,

$$
\begin{aligned}
\Delta_3^{jk}u\big|_{Z_0} &= \sum_{\alpha,\beta=1}^{m} \frac{V_{\alpha\beta}\big|_{Z_0}}{(1-\frac{\delta_{j\alpha}}{2})(1-\frac{\delta_{k\beta}}{2})} \frac{\partial^2 u}{\partial z_{j\alpha}\partial \overline{z}_{k\beta}}\big|_{Z_0} \\
&= \sum_{\alpha,\beta=2}^{m} \frac{V_{\alpha\beta}\big|_{Z_0}}{(1-\frac{\delta_{j\alpha}}{2})(1-\frac{\delta_{k\beta}}{2})} \frac{\partial^2 u}{\partial z_{j\alpha}\partial \overline{z}_{k\beta}}\big|_{Z_0} \\
&= \sum_{\alpha,\beta=2}^{m} \frac{\widetilde{V}_{(\alpha-1)(\beta-1)}\big|_{W_0}}{(1-\frac{\delta_{(j-1)(\delta-1)}}{2})(1-\frac{\delta_{(k-1)(\beta-1)}}{2})} \frac{\partial^2 (u\circ h_1)}{\partial w_{(j-1)(\alpha-1)}\partial \overline{w}_{(k-1)(\beta-1)}}\big|_{W_0} \\
&= \widetilde{\Delta}_3^{(j-1)(k-1)}(u\circ h_1)\big|_{W_0}.
\end{aligned}
\tag{3.13}
$$

Hence

$$
\begin{aligned}
\Delta_3 u\big|_{Z_0} &= \frac{1}{4}\sum_{j,k=1}^{m} V_{jk}\big|_{Z_0}(\Delta_3^{jk}u)\big|_{Z_0} = \frac{1}{4}\sum_{j,k=2}^{m} V_{jk}\big|_{Z_0}(\Delta_3^{jk}u)\big|_{Z_0} \\
&= \frac{1}{4}\sum_{j,k=2}^{m} \widetilde{V}_{(j-1)(k-1)}\big|_{W_0}\widetilde{\Delta}_3^{(j-1)(k-1)}(u\circ h_1)\big|_{W_0}.
\end{aligned}
\tag{3.14}
$$

This implies $\widetilde{\Delta}_3(u\circ h_1)\big|_{W_0} = 0$. Since W_0 is arbitrarily chosen in D_{m-1}^{III}, we conclude $u\circ h_1$ is Bergman-harmonic in D_{m-1}^{III}. $\square$

Proof of Theorem 6 in the type III case: It can be proved by the same argument as in the type I case and by using Lemma 3.6.

We next prove the boundary pluriharmonicity for Bergman-harmonic functions on type III domains.

Proposition 3.4. *(1) If $m \geq 4$ is even and $u \in C^{\frac{m}{2}}(\overline{D_m^{III}})$ is Bergman-harmonic in D_m^{III}, then u is pluriharmonic on every germ of complex manifold in ∂D_m^{III}.*

(2) If $m \geq 3$ is odd and if there exists $\alpha > \frac{1}{2}$ such that $u \in C^{\frac{m-1}{2},\alpha}(\overline{D_m^{III}})$ is Bergman-harmonic in D_m^{III}, then u is pluriharmonic on every germ of complex manifold in ∂D_m^{III}.

Proof. It suffices to prove that u is pluriharmonic on every boundary component of D_m^{III}. Fix $1 \leq k \leq m-1$. Assume $h : D_{m-k}^{III} \to \partial D_m^{II}$ gives a boundary $k-$component. Theorem 6 yields $u\circ h$ is Bergman-harmonic on D_{m-2k}^{II}. By assumption, $u \in C^{\gamma}(\overline{D_m^{II}})$, where $\gamma = \frac{m}{2}$ if m is even, and γ denotes $(\frac{m-1}{2}, \alpha)$ if m is odd. Consequently, since h extends holomorphically across the boundary of D_{m-k}^{III} (This can be seen similarly as in Remark 3.3), we have $u\circ h$ extends $\gamma-$smoothly to a neighborhood of $\overline{D_{m-k}^{III}}$. We then apply Theorem 1.2 (ii), (iii) in [CL] to conclude $u\circ h$ is pluriharmonic in D_{m-k}^{III}. Since k is arbitrarily chosen, we have proved that u is pluriharmonic on every boundary component, and Proposition 3.4 follows. $\square$

Again note every nontrivial complex variety in $E_{m-1} \subset \partial D_m^{III}$ must be of one-dimensional.

Proposition 3.5. *Let $m \geq 2$ and V an open set in $\mathbb{C}_{III}^{\frac{m(m+1)}{2}}$ containing $\overline{D_m^{III}} - E_m$. Assume $u \in C^2(V)$ is Bergman-harmonic in D_m^{III}, then u must be harmonic on every germ of complex curve in $E_{m-1} \subset \partial D_m^{III}$.*

Proof. Copy the proof of Proposition 3.3. □

3.4. **Boundary pluriharmonicity on type IV domains.** We finally prove Theorem 6 for type IV domains and establish boundary pluriharmonicity for Bergman-harmonic functions on type IV domains. Since the book [PS] does not characterize the boundary components of type IV domains, we first give the following description for their boundary components. Recall Δ denotes the unit disk in $\mathbb{C}$.

Lemma 3.7. *(1) Let X_1 be the set of points in ∂D_m^{IV} of the form:*

$$(\frac{\xi+1}{2}, \frac{\xi-1}{2i}, 0, \cdots, 0), \quad \xi \in \Delta, \tag{3.15}$$

Then X_1 gives a boundary component of D_m^{IV}.

(2) Any nontrivial boundary component X of D_m^{IV} can be transformed to X_1 by some automorphism of D_m^{IV}.

Remark 3.4. The component X in (3.15) can be regarded as a boundary $1-$component. It lies entirely in E_1. Each point in E_2 can be regarded as a boundary $2-$component. In particular, Lemma 3.7 says that all nontrivial boundary components must be one-dimensional.

Proof of Lemma 3.7: First one can verify the curve X_1 is contained in the boundary of D_m^{IV}. Indeed, it entirely lies in E_1. Moreover, $\phi(X_1)$ lies entirely in E_1 as well for every $\phi \in \mathrm{Aut}(D_m^{IV})$. Next we prove the following fact.

Claim: Let $g(t), |t| < \epsilon$, be a germ of nontrivial complex curve in ∂D_m^{IV}. It must lie entirely in E_1. Assume h intersects with X_1. Then $g(t)$ is entirely contained in X_1. Consequently, X_1 is a boundary component.

Proof of Claim: First note we must have $g(t) \in E_1$ for all t. Indeed, suppose there is some $|t_0| < \epsilon$ such that $g(t_0) \in E_2$. Then $\|g(t_0)\| = 1$ by (2.1). But we have $\|g(t)\| \leq 1$ for all $|t| < \epsilon$ as h lies in ∂D_m^{IV}. By the maximum principle, $\|g(t)\| = 1$ for all t. This is a contradiction as the sphere in $\mathbb{C}^m$ contains only trivial complex varieties. Hence $|g(t)| < 1$ for all t and the curve g lies entirely in E_1.

By reparametrizing g if necessary, we assume g and X_1 intersect at $Z_0 = g(0) = (\frac{\xi_0+1}{2}, \frac{\xi_0-1}{2i}, 0, \cdots, 0)$ for some $\xi_0 \in \Delta$. Note we can apply an automorphism φ of Δ to make $\varphi(\xi_0) = 0$. Moreover, this automorphism φ of Δ extends to an automorphism Φ of D_m^{IV} that preserves X_1. Indeed, first this φ extends to an automorphism of a maximal polydisc embedded in D_m^{IV} :

$$\Delta^2 \to D_m^{IV} : (\xi, \eta) \to (\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2i}, 0, \cdots, 0).$$

And then note every automorphism of a maximal polydisc extends to an automorphism of D_m^{IV}(cf. [M]).

By the above argument, we can assume $Z_0 = g(0) = (\frac{1}{2}, \frac{i}{2}, 0, \cdots, 0) \in E_1$. Write $g(t) = (g_1(t), \cdots, g_m(t))$ and

$$g_1 = \frac{1}{2} + f_1, \quad g_2 = \frac{i}{2} + f_2.$$

Thus $f_1(0) = f_2(0) = 0$, and $g_i(0) = 0$ for $i \geq 3$. Since g lies in $E_1 \subset \partial D_m^{IV}$, we have

$$1 - 2\|g\|^2 + |gg^t|^2 = 0, \text{ or } \quad 1 - 2\left(|\frac{1}{2} + f_1|^2 + |\frac{i}{2} + f_2|^2 + \sum_{i=3}^{m} |g_i|^2\right) + |gg^t|^2 = 0. \tag{3.16}$$

Write $G = \frac{gg^t}{\sqrt{2}} = \frac{\sum_{i=1}^m g_i^2}{\sqrt{2}}$. Note $G(0) = 0$. Moreover, (3.16) yields

$$-|G|^2 + |f_1|^2 + |f_2|^2 + \sum_{i=3}^{m} |g_i|^2 + \mathrm{Re}(f_1 - if_2) = 0 \text{ for } |t| < \epsilon.$$

Equating pure and mixed terms in z and $\overline{z}$, we get

$$\mathrm{Re}(f_1 - if_2) = 0; \tag{3.17}$$

$$|G|^2 = |f_1|^2 + |f_2|^2 + \sum_{i=3}^{m} |g_i|^2. \tag{3.18}$$

By D'Angelo's lemma [D], there is an $m \times m$ unitary matrix U such that

$$(f_1, f_2, g_3, \cdots, g_m) = (G, 0, \cdots, 0)U.$$

Writing the first row of U as $\mathbf{u} = (u_1, \cdots, u_m)$, we have

$$(f_1, f_2, g_3, \cdots, g_m) = G(u_1, \cdots, u_m). \tag{3.19}$$

We claim that G is not identically zero. Otherwise, by the above equation, f_1, f_2 and all g_i's are constant. This contradicts with the assumption that g is nontrivial.

Now (3.17) and (3.19) yield

$$\mathrm{Re}\left((u_1 - iu_2)G\right) = 0.$$

We apply open mapping theorem to G to obtain

$$u_1 = iu_2. \tag{3.20}$$

On the other hand, we have

$$\sqrt{2}G = \sum_{i=1}^{m} g_i^2 = (\frac{1}{2} + f_1)^2 + (\frac{i}{2} + f_2)^2 + g_3^2 + \cdots + g_m^2.$$

Combining this with (3.19) and (3.20), we obtain

$$\sqrt{2}G = 2u_1 G + (u_1^2 + \cdots + u_m^2)G^2.$$

Applying again open mapping theorem to G, we get

$$2u_1 = \sqrt{2}, \quad u_1^2 + \cdots + u_m^2 = 0.$$

Thus $u_1 = \frac{\sqrt{2}}{2}$ and $u_2 = -i\frac{\sqrt{2}}{2}$. Consequently, as $\mathbf{u}$ is a unit vector, we have $u_3 = \cdots = u_m = 0$. We thus conclude by (3.19) that

$$g = (\frac{1}{2} + \frac{\sqrt{2}}{2}G, \frac{i}{2} - i\frac{\sqrt{2}}{2}G, 0, \cdots, 0).$$

We note $|G| < \frac{\sqrt{2}}{2}$ for all t. Indeed,

$$|G|^2 = \frac{|gg^t|^2}{2} \leq \frac{\|g\|^4}{2} < \frac{1}{2}.$$

The last inequality comes from the fact that g lies in $E_1 \subset \partial D_m^{IV}$. Finally it follows that the complex curve g lies entirely in X_1. Since g is any arbitrary complex curve intersecting X_1, we conclude X_1 is a boundary component. This proves the claim. ■

Now let f be any germ of nontrivial complex curve in ∂D_m^{IV}. By the claim above , it must lies in E_1. As $\mathrm{Aut}(D_m^{IV})$ acts transitively on E_1, there is some $\phi \in \mathrm{Aut}(D_m^{IV})$ such that $\phi^{-1}(f)$ intersects X_1. It follows from the above claim that f lies entirely in $\phi(X_1)$. Hence we conclude that $\{\phi(X_1) : \phi \in \mathrm{Aut}(D_m^{IV})\}$ are boundary components of D_m^{IV}, and are the only nontrivial boundary components. This proves Lemma 3.7. ■

In light of Proposition 3.7, we will call $h_1 : \Delta \to \partial D_m^{IV}$ the standard boundary $1-$component where h_1 is given by (3.15).

Lemma 3.8. *Let V be an open subset of $\mathbb{C}^m$ containing $\overline{D_m^{IV}} - E_2$ and $u \in C^2(V)$ be Bergman-harmonic in D_m^{IV}. Then $u \circ h_1$ is Bergman-harmonic (equivalently harmonic in this case) on Δ.*

Proof. Let Δ_4 be the Hua operator on the type IV domain D_m^{IV} as in (2.7). As in the other type cases, we have $\Delta_4 u = 0$ in V by the Bergman-harmonicity and boundary regularity of u. Thus if we fix $\xi_0 \in \Delta$ and write $Z_0 = h_1(\xi_0) = (\frac{\xi_0+1}{2}, \frac{\xi_0-1}{2i}, 0, \cdots, 0)$, we have $\Delta_4 u = 0$ at $Z = Z_0$. We next compute $\Delta_4 u$ in term of (2.7). Let $s(Z), r(Z)$ be as in (2.6). Recall $Z = (z_1, \cdots, z_m)$. Note

$$s(Z_0) = Z_0 Z_0^t = (\frac{\xi_0+1}{2})^2 + (\frac{\xi_0-1}{2i})^2 = \xi_0;$$

$$\|Z_0\|^2 = Z_0 \overline{Z}_0^t = |\frac{\xi_0+1}{2}|^2 + |\frac{\xi_0-1}{2i}|^2 = \frac{1}{2}(|\xi_0|^2+1).$$

Thus

$$r(Z_0) = 1 - 2\|Z_0\|^2 + |s(Z_0)|^2 = 0.$$

Consequently, by (2.7)

$$\Delta_4 u\big|_{Z_0} = 2\sum_{j,k=1}^{n} (\overline{z}_j - s(\overline{Z})z_j)\big|_{Z_0} (z_k - s(Z)\overline{z}_k)\big|_{Z_0} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\big|_{Z_0} = 2\sum_{j,k=1}^{2} (\overline{z}_j - s(\overline{Z})z_j)\big|_{Z_0} (z_k - s(Z)\overline{z}_k)\big|_{Z_0} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\big|_{Z_0}. \tag{3.21}$$

Writing

$$A_1 = (z_1 - s(Z)\overline{z}_1)\big|_{Z_0} = \frac{1-|\xi_0|^2}{2}, A_2(Z) = (z_2 - s(Z)\overline{z}_2)\big|_{Z_0} = \frac{|\xi_0|^2-1}{2i},$$

we have

$$\begin{aligned}\Delta_4 u\big|_{Z_0} &= 2\left(|A_1|^2\frac{\partial^2 u}{\partial z_1\partial\overline{z}_1}\big|_{Z_0}+\overline{A}_1A_2\frac{\partial^2 u}{\partial z_1\partial\overline{z}_2}\big|_{Z_0}+A_1\overline{A}_2\frac{\partial^2 u}{\partial z_2\partial\overline{z}_1}\big|_{Z_0}+|A_2|^2\frac{\partial^2 u}{\partial z_2\partial\overline{z}_2}\big|_{Z_0}\right)\\ &= \frac{(1-|\xi_0|^2)^2}{2}\left(\frac{\partial^2 u}{\partial z_1\partial\overline{z}_1}\big|_{Z_0}+i\frac{\partial^2 u}{\partial z_1\partial\overline{z}_2}\big|_{Z_0}-i\frac{\partial^2 u}{\partial z_2\partial\overline{z}_1}\big|_{Z_0}+\frac{\partial^2 u}{\partial z_2\partial\overline{z}_2}\big|_{Z_0}\right)=0.\end{aligned} \tag{3.22}$$

On the other hand, by the chain rule, we obtain,

$$\frac{\partial^2(u\circ h_1)}{\partial\xi\partial\overline{\xi}}\big|_{\xi_0}=\frac{1}{4}\left(\frac{\partial^2 u}{\partial z_1\partial\overline{z}_1}\big|_{Z_0}+i\frac{\partial^2 u}{\partial z_1\partial\overline{z}_2}\big|_{Z_0}-i\frac{\partial^2 u}{\partial z_2\partial\overline{z}_1}\big|_{Z_0}+\frac{\partial^2 u}{\partial z_2\partial\overline{z}_2}\right). \tag{3.23}$$

This is a multiple of $\Delta_4 u(Z_0)$ by (3.22), and thus equals zero. Since ξ_0 is arbitrarily chosen in Δ, we conclude $u\circ h_1$ is harmonic in Δ. This establishes Lemma 3.8. □

Proof of Theorem 6 for the type IV case: Let $h:\Delta\to\partial D_m^{IV}$ be some nontrivial boundary component (i.e., boundary $1-$component in this case). By Proposition 3.1, $h=\psi\circ h_1\circ\phi$ for some $\phi\in\mathrm{Aut}(\Delta)$ and $\psi\in\mathrm{Aut}(D_m^{IV})$. By Lemma 3.8, $u\circ h_1$ is harmonic in Δ. It then follows from the same argument as in the type I case that $u\circ h$ is Bergman-harmonic (i.e., harmonic) on Δ. This proves Theorem 6 in the type IV case. ■

Theorem 6 immediately implies the following fact.

Proposition 3.6. *Let $m\geq 2$ and V an open subset of $\mathbb{C}^m$ containing $\overline{D_m^{IV}}-E_2$. Assume $u\in C^2(V)$ is Bergman-harmonic on D_m^{IV}. Then u must be harmonic on every germ of complex curve in ∂D_m^{IV}.*

Inspired by Propositions 3.3, 3.5 and 3.6, an interesting and natural question asks whether Theorem 3 still holds if we only assume the boundary regularity $u\in C^l(V)$ where V as in Theorem 6. Similar questions can be asked for Proposition 3.2, 3.4.

Proof of Theorem 4: Theorem 4 follows from Propositions 3.3, 3.5, 3.6.

To end Section 3, we remark that it can be seen that, from the proof of Theorem 6 in each case, the following stronger version holds.

Proposition 3.7. *Let $D\subset\mathbb{C}^m$ be a classical domain and V an open subset of $\mathbb{C}^m$ containing the set $D\cup E_k$, where E_k is k^{th} boundary orbit of D. Assume $u\in C^2(V)$ is Bergman-harmonic function in D. Then u is Bergman-harmonic on every boundary component of D in E_k. That is, for every $h:D'\to D$ which gives a boundary component of D in E_k, it holds that $u\circ h$ is Bergman-harmonic in D'.*

4. A new characterization of pluriharmonicity

4.1. **Preliminary.** Let Ω be a domain in $\mathbb{C}^n$ and u a C^2-smooth function in Ω. A simple and well-known fact states that u is pluriharmonic if and only if h is harmonic on $L\cap\Omega$ for every complex affine line L. In this section, we establish a new characterization of pluriharmonicity by using the special geometric structure of bounded symmetric domains, and will illustrate the aforementioned fact is indeed the most special case of our result.

To explain our result, we first recall the notion of minimal disks of bounded symmetric domains. Recall a bounded symmetric domain is of rank one if and only if it is biholomorphic to the unit ball. A bounded symmetric domain D of high rank has different sectional curvatures in general along different complex directions. The minimal disks are a special class of complex geodesic curves whose unit holomorphic tangent vectors realize the minimum of the holomorphic sectional curvature of D. The minimal disks can be also interpreted in a more algebraic way. Let X be the compact dual of D. Recall D can be canonically embedded into X as an open subset by the Borel embedding. Under this embedding, a minimal disk of D extends to a minimal rational curve on X, which is defined as a free rational curve of minimal degree among all free rational curves. In particular, when D is the unit ball $\mathbb{B}^n$, the intersection of every complex affine line with $\mathbb{B}^n$ is a minimal disk. Fix $x \in D$. The holomorphic tangent vector at x of a minimal disk passing through x is called a minimal rational tangent. The variety of minimal rational tangents, alias VMRT, at x of D(or of its compact dual X) is defined by

$$\mathcal{C}_x D := \{[\eta] \in \mathbb{P}T_x D : 0 \neq \eta \in \mathbb{C}T_x \Delta_x \text{ for some minimal disk } \Delta_x \text{ passing through } x\}.$$

The VMRT $\mathcal{C}_x D$ can be equipped with some natural structure to be a complex manifold. The following table taken from [M] describes VMRTs for all types of compact Hermitian symmetric spaces. The last column describes the embedding of $\mathcal{C}_x D$ into $\mathbb{P}T_x D$.

Type	G	K	$X \approx G/K$	$\mathcal{C}_x D = \mathcal{C}_x X$	embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2, n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$Spin(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

The theory of VMRT was introduced by Hwang-Mok [HM] in the late 90's. It plays a fundamental role in studying the geometric theory of Hermitian symmetric spaces and more generally uniruled projective manifolds, as well as related topics, such as mapping problems between Hermitian symmetric spaces.

The main result in this section is Theorem 5 mentioned in the introduction. This work draws inspiration from two very interesting papers by Ng [Ng] and by Chen-Li [CL].

Theorem. *(Theorem 5 in Section 1) Let D be an irreducible bounded symmetric domain and $u \in C^2(D)$ a real-valued function. Then u is pluriharmonic if and only if u is harmonic on every minimal disk of D.*

Remark 4.1. When D is the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, the theorem is reduced to the simple fact that u is pluriharmonic if and only if it is harmonic along every complex affine line.

The theorem fails if D is reducible, as can be seen from easy examples. For instance, let $u(z,w) = \mathrm{Re}(z\overline{w})$ on the bidisk $\Delta^2 \subset \mathbb{C}^2$. Then u is harmonic on every minimal disk of Δ^2, but is not pluriharmonic in Δ^2. However, we have the following modified version for reducible domains.

Theorem 7. *Let $D = D_1 \times \cdots \times D_m \subset \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_m}$ be a bounded symmetric domain, where $D_j^{'}s$ are irreducible factors. Assume for all choices of minimal disks $\Delta_j \subset D_j, 1 \leq j \leq m$, we have u is pluriharmonic on $\Delta_1 \times \cdots \times \Delta_m$. Then u is pluriharmonic in D.*

4.2. **Proofs of the characterization.** We give proofs for Theorem 5 and 7 in this section. To prove Theorem 5, we first establish the following results.

Let h be a C^2-smooth function defined on some open subset of $\mathbb{C}^n$. Writing $z = (z_1, \cdots, z_n)$ for the coordinates of $\mathbb{C}^n$, recall the complex Hessian of h is the $n \times n$ Hermitian matrix $\left(\frac{\partial^2 h}{\partial z_i \partial \overline{z}_j}\right)_{1 \leq i,j \leq n}$.

Proposition 4.1. *Let $D \subset \mathbb{C}^{N+1}$ be an irreducible bounded symmetric domain and $u \in C^2(D)$ a real-valued function. Fix $x \in D$. Assume u is harmonic on every minimal disk passing through x. Then the complex Hessian of u vanishes at x.*

Proof. Let $H(v,v)$ be a Hermitian form in $v \in \mathbb{C}^{N+1}$. Write the signature of H as (a,b,c), i.e. (the Hermitan matrix corresponding to) H has a positive eigenvalues, b negative eigenvalues, and c zero eigenvalues, where $a+b+c = N+1$. Define the real hyperquadric in $\mathbb{P}^N$:

$$\mathcal{Q} = \{[v] \in \mathbb{P}^N : H(v,v) = 0\}.$$

Then, inspired by [Ng], we have the following lemma:

Lemma 4.1. *Let $m = N - \max\{a,b\}$. Then the maximal compact complex analytic subvarieties in $\mathcal{Q}$ are $m-$dimensional projective linear subspaces. Moreover, any germ of complex submanifold in $\mathcal{Q}$ is contained in one of these projective linear subspaces.*

Proof of Lemma 4.1: It basically follows from Lemma 2.4 in [Ng]. For the self-containedness, we sketch the proof here. By the assumption on the signature of H, there exists $U \in U(N+1)$, such that

$$H(vU, vU) = v \ \mathrm{diag}(\lambda_1, \cdots, \lambda_a, \mu_1, \cdots, \mu_b, 0, \cdots, 0) \ \overline{v}^t,$$

for some $\lambda_i > 0, 1 \leq i \leq a$ and $\mu_j < 0, 1 \leq j \leq b$. Thus by composing with a linear map $v \to vP$ for some appropriate matrix $P \in GL(N+1, \mathbb{C})$, we can assume H takes the form:

$$H(v,v) = v \ \mathrm{diag}(1, \cdots, 1, -1, \cdots, -1, 0, \cdots, 0) \ \overline{v}^t,$$

where the first a diagonal elements are 1, the next b diagonal elements are -1, and the rest are 0.

We first consider the case when a, b are both positive. Without loss of generality, assume $a \geq b$. Then $m = b + c - 1$. Write the homogeneous coordinates in $\mathbb{P}^N$ as $[z] =$

$[z_1, \cdots, z_a, z_{a+1}, \cdots, z_{a+b}, z_{a+b+1}, \cdots, z_{N+1}]$. Then $\mathcal{Q}$ is defined by

$$\sum_{i=1}^{a} |z_i|^2 = \sum_{j=1}^{b} |z_{a+j}|^2.$$

Write $\mathbb{C}^{m\times n}$ for the set of $m\times n$ matrices with elements in $\mathbb{C}$ and pick any $A \in \mathbb{C}^{b\times a}$ with $A\overline{A}^t = I_b$, where I_b is the $b\times b$ identity matrix. Then the following $(b+c)-$dimensional projective linear subspace of $\mathbb{P}^N$ lies in $\mathcal{Q}$:

$$\{[z] \in \mathbb{P}^N : [z_1, \cdots, z_a] = [z_{a+1}, \cdots, z_{a+b}, z_{a+b+1}, \cdots, z_{N+1}]B\}. \tag{4.1}$$

Here the $(b+c)\times a$ matrix $B = \begin{bmatrix} A \\ \mathbf{0}_{c\times a} \end{bmatrix}$ and $\mathbf{0}_{c\times a}$ denotes the $c\times a$ zero matrix.

On the other hand, given a germ of complex manifold $g : V \to \mathcal{Q}$, where V is a small open set in some $\mathbb{C}^l$. By shrinking V if necessary, we can assume $g(V)$ is contained in a Euclidean affine cell of $\mathbb{P}^N$, say $U_1 := \{[z_1, \cdots, z_{N+1}] : z_1 \neq 0\}$. Then in terms of inhomogeneous coordinates, $g = (g_2, \cdots, g_{N+1})$ satisfies the equation:

$$1 + \sum_{i=2}^{a} |g_i|^2 = \sum_{j=1}^{b} |g_{a+j}|^2.$$

By D'Angelo's lemma [D], there exists $A \in M(b, a; \mathbb{C})$ with $A\overline{A}^t = I_b$ such that

$$(1, g_2, \cdots, g_a) = (g_{a+1}, \cdots, g_{a+b})A.$$

Thus such a germ of complex submanifold must be contained in one of the $(b+c)-$dimensional projective linear subspace as described in (4.1). Hence those $\mathbb{P}^{b+c}$ are maximal complex subvarieties in $\mathcal{Q}$.

When $a = 0$ or $b = 0$, we have $m = c-1$. It is clear that in this case any germ of complex submanifold is contained in the following $(c-1)-$dimensional projective linear subspace:

$$\{[0, \cdots, 0, z_{a+b+1}, \cdots, z_{N+1}] : [z_{a+b+1}, \cdots, z_{N+1}] \in \mathbb{P}^{c-1}\} \subset \mathcal{Q}.$$

This finishes the proof of Lemma 4.1. ■

As a consequence of Lemma 4.1, we have the following result. Identify $\mathbb{P}T_xD$ with $\mathbb{P}^N$ in the natural way.

Lemma 4.2. *Assume $H(v, v) = 0$ for all $[v] \in \mathcal{C}_xD \subset \mathbb{P}^N$. Then the Hermitian form $H(\cdot,\cdot)$ is identically zero in $\mathbb{C}^{N+1}$.*

Proof of Lemma 4.2: Recall $H(\cdot,\cdot)$ has signature (a, b, c). It suffices to prove $c = N+1$. By assumption, $\mathcal{C}_xD \subset \mathcal{Q} \subset \mathbb{P}^N$. It follows from Lemma 4.1 that $\mathcal{C}_x$ is contained in some $m-$dimensional projective linear subspace of $\mathbb{P}^N$, where $m = N - \max\{a, b\}$. We conclude from the following claim that $m = N$, i.e., $a = b = 0$.

Claim: $\mathcal{C}_xD$ is not contained in any lower dimensional projective linear subspace of $\mathbb{P}^N$.

Proof of Claim: This fact is known to experts, and can be easily seen from the table in subsection 4.1. For instance, in the type I domain case $D^I_{p,q} = \{Z : \mathbb{C}^{p\times q} : I_p - Z\overline{Z}^t > 0\}$, the VMRT $\mathcal{C}_x D^I_{p,q}$ is the image of the Segre map $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \to \mathbb{P}^{pq-1}$. Recall the Segre map is given by

$$([z_1, \cdots, z_p], [w_1, \cdots, w_q]) \to [z_i w_j]_{1\leq i\leq p, 1\leq j\leq q}.$$

The claim can be seen from the linear independence of the quadratic monomials $z_i w_j'$s. In the type IV and V case, the embedding of VMRT $\mathcal{C}_x D$ into $\mathbb{P}^N$ is the minimal embedding given by $\mathcal{O}(1)$. Write $\mathcal{S}_{E_6}, \mathcal{S}_{E_7}$ for the Hermitian symmetric spaces of compact type that correspond to E_6, E_7. In the type VI case in the table, we have the VMRT of $\mathcal{S}_{E_7}$ satisfies $\mathcal{C}_x\mathcal{S}_{E_7} \simeq \mathcal{S}_{E_6}$ and the embedding of $\mathcal{C}_x\mathcal{S}_{E_7}$ into $\mathbb{P}T_x\mathcal{S}_{E_7}$ is indeed the minimal embedding $\mathcal{S}_{E_6} \to \mathbb{P}^{26}$ (cf. [FHX] for the explicit formula of this embedding). The claim for the remaining cases in the table can be verified similarly. ■

This establishes Lemma 4.2. ■

We now continue to prove Proposition 4.1. Write $H_x(\cdot, \cdot)$ for the Hermitian form associated to the complex Hessian of u at x. Fix $[v] \in \mathcal{C}_x D$. Then there exists a minimal disk Δ_x of D such that v spans $\mathbb{C}T_x\Delta_x$. By the assumption that u is harmonic on Δ_x, we have $H_x(v, v) = 0$. Note $[v]$ is arbitrary in $\mathcal{C}_x D$. Then it follows from Lemma 4.2 that $H_x(\cdot, \cdot)$ is identically zero in $\mathbb{C}^{N+1}$. This proves Propopsition 4.1.

□

Proof of Theorem 5: Theorem 5 is now a consequence of Proposition 4.1.

Remark 4.2. It is clear from Proposition 4.1 that the following stronger versions of Theorem 5 holds: Let U be an open subset of D. Let $u \in C^2(U)$ be a real-valued function and S a dense subset of U. Assume for every $x \in S$ and every minimal disk Δ_x passing through x, u is harmonic on $\Delta_x \cap U$. Then u is pluriharmonic in U.

Next we give a proof of Theorem 7.

Proof of Theorem 7: Fix a point $x = (x_1, \cdots, x_m) \in D$ where $x_k \in D_k, 1 \leq k \leq m$. Write $\mathcal{C}_{x_k} D_k \subset \mathbb{P}T_{x_k} D_k \simeq \mathbb{P}^{N_k - 1}$. Write $N = N_1 + \cdots + N_m$ and the homogeneous coordinates in $\mathbb{P}^{N-1}$ as:

$$[Z] = [Z_1, \cdots, Z_m]$$

where Z_k denotes the coordinates of $\mathbb{C}^{N_k}, 1 \leq k \leq m$. We write the defining function of $\mathcal{C}_{x_k} D_k$, which is a projective variety in $\mathbb{P}^{N_k-1}$, as

$$P_k(Z_k) = 0 \text{ with } [Z_k] \in \mathbb{P}^{N_k-1}$$

for some system of homogeneous polynomials P_k. Define a projective variety V of $\mathbb{P}T_x D \simeq \mathbb{P}^{N-1}$ by

$$V := \{[Z] = [Z_1, \cdots, Z_m] : P_k(Z_k) = 0 \text{ for all } 1 \leq k \leq m\}.$$

Write $H_x(\cdot, \cdot)$ for the Hermitian form associated to the complex Hessian of u. By the assumption of Theorem 7, we have

$$H_x(Z, Z) = 0 \text{ for all } [Z] \in V.$$

As before, write $\mathcal{Q}$ for the zero locus of $H_x(\cdot,\cdot)$ in $\mathbb{P}^{N-1}$: $\mathcal{Q} = \{[Z] \in \mathbb{P}^{N-1} : H_x(Z,Z) = 0\}$. Then $V \subset \mathcal{Q}$. We will show that V is not contained in any lower dimensional projective linear subspace of $\mathbb{P}^{N-1}$. Then it follows from Lemma 4.1 that $H_x(\cdot,\cdot)$ has signature $(0,0,N)$, i.e., $H_x(\cdot,\cdot)$ is identically zero in $\mathbb{C}^N$.

We prove by contradiction. Suppose V is contained in some $(N-2)-$dimensional projective linear subspace; call it L. Define for each $1 \leq k \leq m$,

$$L_k := \{[Z] \in \mathbb{P}^{N-1} : Z_l = 0 \text{ if } l \neq k\} \simeq \mathbb{P}^{N_K - 1}.$$

Then there exists some $1 \leq k_0 \leq m$ such that $L \cap L_{k_0} \neq L_{k_0}$(Otherwise $L \simeq \mathbb{P}^{N-1}$). But this implies $C_{x_{k_0}} D_{k_0} \simeq V \cap L_{k_0}$ is contained in a lower-dimensional projective linear subspace $L \cap L_{k_0}$ in $\mathbb{P}^{N_{k_0}-1}$. This contradicts with the claim in the proof of Theorem 5.

We have thus proved $H_x(\cdot,\cdot)$ is identically zero in $\mathbb{C}^N$. Since x is arbitrary in D, we conclude H is pluriharmonic in D.

4.3. **Some applications.** We expect Theorem 5 to be useful in the future study of Bergman-harmonic functions on bounded symmetric domains (See the end of Section 5). There are also some other possible applications. First Theorem 5 has the following geometric formulation.

Proposition 4.2. *Let g_1, g_2 be two Kähler metrics on an irreducible bounded symmetric domain D. Then $g_1 = g_2$ on D if and only if they have the same induced metric on every minimal disk.*

Proof. The "only if" part is trivial. To prove the converse direction, fix $x \in D$. First by the Kählerness assumption, $g_i = \partial\overline{\partial}\rho_i, 1 \leq i \leq 2$, for some C^2-functions $\rho_1(Z,\overline{Z}), \rho_2(Z,\overline{Z})$ in some neighborhood U of x in D. Let $h : \Delta \to D$ be any minimal disk in D that has a nonempty intersection with U. By assumption, the two metrics g_1, g_2 induce the same metric on Δ. This implies

$$\partial\overline{\partial}(\rho_1 \circ h) = \partial\overline{\partial}(\rho_2 \circ h) \text{ on } \Delta \cap U.$$

Here $\rho_i \circ h = \rho_i(h, \overline{h})$. As a consequence, $\rho_1 - \rho_2$ is harmonic on $\Delta \cap D$ for every minimal disk Δ. Then Theorem 1 (Remark 4.2) yields that $\rho_1 - \rho_2$ is pluriharmonic in U. Hence $g_1 = g_2$ on U, and thus on D by the arbitrary choice of x. □

Let $(M_1, \omega_1), (M_2, \omega_2)$ be two Kähler manifolds and $F : M_1 \to M_2$ be a holomorphic map. We say F is isometric if $F^*(\omega_2) = \omega_1$.

Corollary 4.1. *Let (D, ω_D) be an irreducible bounded symmetric domain equipped with the Bergman metric ω_D. Let (M, ω) be a Kähler manifold and F a holomorphic map from D to M. Then F is isometric if and only if F maps each minimal disk isometrically into M.*

Proof. We only need to prove the "if" part. Note $F^*(\omega)$ defines a Kähler metric on D. By assumption, F maps all minimal disks isomerically to M. This implies that ω_D and $F^*(\omega)$ induce the same metric on every minimal disk. By Proposition 4.2, $F^*(\omega) = \omega_D$. □

5. A further remark on Bergman-harmonic functions on type I domains

In this section, we will prove a characterization result for Bergman-harmonic functions on type I domains, using the ideas in Section 3. To explain our result, we first recall the definition of invariantly geodesic subspace of a bounded symmetric domain (See [MT]). For our purpose, we will formulate it for type I domains only. Let $S \subset D^I_{p,q}$ be a complex submanifold and regard $D^I_{p,q}$ as an open subset of $G_{p,q}$, the Grassmannian of $p-$planes in $\mathbb{C}^{p+q}$. Note the special group $\mathrm{SL}(p+q,\mathbb{C})$, i.e., the set of $(p+q)\times(p+q)$ matrices with determinant 1, acts naturally on $G_{p,q}$. We say S is an invariantly geodesic subspace of $D^I_{p,q}$ if for every $g \in \mathrm{SL}(p+q)$ with $g(S)\cap D^I_{p,q} \neq \emptyset$, the submanifold $g(S)\cap D^I_{p,q}$ is totally geodesic with respect to the Bergman metric of $D^I_{p,q}$.

By the work of Mok-Tsai [MT], every invariantly geodesic subspace S of $D^I_{p,q}$, up to the action of automorphisms of $D^I_{p,q}$, equivalent to a submanifold given by the image of the map $h_0 : D^I_{r,s} \to D^I_{p,q}$ for some $1 \leq r \leq p, 1 \leq s \leq q$. Here

$$h_0(Z) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z \end{pmatrix}, \quad Z \in D^I_{r,s}.$$

Such S is called an (invariantly geodesic) $(r,s)-$subspace.

In light of Mok-Tsai [MT], we make the following definition.

Definition 5.1. Let $1 \leq r \leq p, 1 \leq s \leq q$.

(1) We say $h : D^I_{r,s} \to D^I_{p,q}$ gives a $(r,s)-$subspace if the image $h(D^I_{r,s})$ is a $(r,s)-$subspace and h is biholomorphism from $D^I_{r,s}$ to $h(D^I_{r,s})$.
(2) Let $u \in C^2(D^I_{p,q})$. We say u is Bergman-harmonic on the $(r,s)-$subspace $h(D^I_{r,s})$ if $u \circ h$ is Bergman-harmonic in $D^I_{r,s}$.

Remark 5.1. The notion in part (2) is independent of the choice of h. Indeed, suppose $\hat{h} : D^I_{r,s} \to D^I_{p,q}$ gives the same $(r,s)-$subspace, i.e., $\hat{h}(D^I_{r,s}) = h(D^I_{r,s})$. Then we must have $\hat{h} = h \circ \phi$ for some $\phi \in \mathrm{Aut}(D^I_{p,q})$. The biholomorphic invariance of Bergman-harmonic functions implies $u \circ \hat{h}$ is Bergman-harmonic on $D^I_{r,s}$ if and only if $u \circ h$ is so.

Proposition 5.1. *Let $1 \leq p \leq q$. Fix $1 \leq k \leq p$. Then $u \in C^2(D^I_{p,q}) \cap C(\overline{D^I_{p,q}})$ is Bergman-harmonic in $D^I_{p,q}$ if and only if u is Bergman-harmonic on every invariantly geodesic $(k,q)-$subspace.*

Proof. We will first prove for the case $k = 1$. That is,

Lemma 5.1. *Let $u \in C^2(D^I_{p,q}) \cap C(\overline{D^I_{p,q}})$. Then u is Bergman-harmonic in $D^I_{p,q}$ if and only if u is Bergman-harmonic on every $(1,q)-$subspace of $D^I_{p,q}$.*

Proof of Lemma 5.1: Assume u is Bergman-harmonic in $D^I_{p,q}$ and first prove the "only if" part. We start with the standard $(1,q)-$subspace, i.e., the subspace given by $h_1 : \mathbb{B}^q \to D^I_{p,q}$, where h_1 is defined as

$$h_1(\xi) = \begin{pmatrix} \xi \\ \mathbf{0} \\ \cdots \\ \mathbf{0} \end{pmatrix}, \quad \xi = (\xi_1, \cdots, \xi_q) \in \mathbb{B}^q.$$

Write $\Delta_{\mathbb{B}^q}$ for the Bergman-Laplacian of $\mathbb{B}^q$ and write $Z = (z_{ij})_{1\leq i\leq p, 1\leq j\leq q}$ for the coordinates in $\mathbb{C}^{p\times q}$. We have

$$\begin{aligned}\Delta_{\mathbb{B}^q}(u\circ h_1) &= (1-\|\xi\|^2)\sum_{\alpha,\beta=1}^{q}(\delta_{\alpha\beta}-\xi_\alpha\overline{\xi}_\beta)\frac{\partial^2(u\circ h_1)}{\partial\xi_\alpha\partial\overline{\xi}_\beta} \\ &= (1-\|\xi\|^2)\sum_{\alpha,\beta=1}^{q}(\delta_{\alpha\beta}-\sum_{l=1}^{p}z_{l\alpha}\overline{z}_{l\beta})\big|_{h_1(\xi)}\frac{\partial^2 u}{\partial z_{1\alpha}\partial\overline{z}_{1\beta}}\big|_{h_1(\xi)} \\ &= (1-\|\xi\|^2)\Delta_1^{11}u\big|_{h_1(\xi)}\end{aligned} \tag{5.1}$$

This is zero by [H] (See Theorem 2.2 (i) in [CL]). Thus u is Bergman-harmonic on the standard $(1,q)-$subspace. Now assume $h:\mathbb{B}^q\to D^I_{p,q}$ give any $(1,q)-$subspace. Recall all $(1,q)-$subspaces of $D^I_{p,q}$ are equivalent up to the action of automorphisms of $D^I_{p,q}$ by [MT]. This implies $h=\psi\circ h_1\circ\phi$ for some $\phi\in\mathrm{Aut}(\mathbb{B}^q), \psi\in\mathrm{Aut}(D^I_{p,q})$. We claim that $u\circ h$ is Bergman-harmonic on $\mathbb{B}^q$. Indeed, note $u\circ\psi$ is also Bergman-harmonic by the biholomorphic invariance of Bergman-harmonic functions. By the above argument, $u\circ\psi\circ h_1$ is Bergman-harmonic in $\mathbb{B}^q$. Consequently, $u\circ h = u\circ\psi\circ h_1\circ\phi$ is Bergman-harmonic on $\mathbb{B}^q$ as $\phi\in\mathrm{Aut}(\mathbb{B}^q)$. This proves the "only if "part.

We next prove the converse. Assume u is Bergman-harmonic on every $(1,q)-$subspace. Then so is $u\circ\psi$ for every $\psi\in\mathrm{Aut}(D^I_{p,q})$. This is because $\psi\circ h$ also gives a $(1,q)-$subspace if $h:\mathbb{B}^q\to D^I_{p,q}$ gives a $(1,q)-$subspace.

Set for each $1\leq i\leq p, h_i:\mathbb{B}^q\to D^I_{p,q}$ to be

$$h_i(\xi) = \begin{pmatrix} \mathbf{0} \\ \cdots \\ \mathbf{0} \\ \xi \\ \mathbf{0} \\ \cdots \\ \mathbf{0} \end{pmatrix},$$

where ξ is at the i^{th} row. Clearly each h_i gives a $(1,q)-$subspace of $D^I_{p,q}$. By assumption, we have $\Delta_{\mathbb{B}^q}(u\circ h_i)=0$. Consequently,

$$(1-\|\xi\|^2)\sum_{1\leq\alpha,\beta\leq q}(\delta_{\alpha\beta}-\xi_\alpha\overline{\xi}_\beta)\frac{\partial^2(u\circ h_i)}{\partial\xi_\alpha\partial\overline{\xi}_\beta}=0.$$

Evaluate at $\xi = 0$ to get,

$$\sum_{1 \le \alpha \le q} \frac{\partial^2 (u \circ h_i)}{\partial \xi_\alpha \partial \bar{\xi}_\alpha}\Big|_{\xi=0} = \sum_{j=1}^{q} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{ij}}\Big|_{Z=0} = 0.$$

Taking sum over $1 \le i \le p$, we conclude that

$$\Delta u(0) = \sum_{i=1}^{p} \sum_{j=1}^{q} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{ij}}\Big|_{Z=0} = 0,$$

where Δ is the standard Laplacian in $\mathbb{C}^{p \times q}$. Applying this conclusion to $u \circ \psi$, we get

$$\Delta(u \circ \psi)(0) = 0 \text{ for any } \psi \in \mathrm{Aut}(D^I_{p,q}). \tag{5.2}$$

Fix $Z_0 \in D^I_{p,q}$. There is $\widetilde{\psi} \in \mathrm{Aut}(D^I_{p,q})$ such that $\widetilde{\psi}(0) = Z_0$. By the biholomorphic invariance of Bergman laplacian, we have $\Delta_1(u \circ \widetilde{\psi}) = (\Delta_1 u) \circ \widetilde{\psi}$. Since Δ_1 and Δ concide at origin, we have by (5.2),

$$0 = \Delta(u \circ \widetilde{\psi})(0) = \Delta_1(u \circ \widetilde{\psi})(0) = (\Delta_1 u) \circ \widetilde{\psi}(0) = \Delta_1 u(Z_0).$$

Since Z_0 is arbitrary, we obtain $\Delta_1 u = 0$ in $D^I_{p,q}$ and u is Bergman-harmonic on $D^I_{p,q}$. This finishes the proof of Lemma 5.1. ■

Fix $1 \le k \le p$ and a $(k,q)-$subspace D of $D^I_{p,q}$. Note every $(1,q)-$subspace of D is also a $(1,q)-$subspace of $D^I_{p,q}$. Conversely, every $(1,q)-$subspace of $D^I_{p,q}$ is a $(1,q)-$subspace of some $(k,q)-$subspace of $D^I_{p,q}$. Thus Proposition 5.1 holds for every $1 \le k \le p$. □

Remark 5.2. The analog of Proposition 5.1 may not hold for other types of classical domains, due to the different structures of their invariantly gedesic subspaces.

At the end, we remark that the proof of Theorem 2 (i.e., Theorem 1.2 (i) in [CL]) has a geometric formulation as follows: Let u be as in Theorem 2. By Proposition 5.1, u is Bergman-harmonic on every $(1,q)-$space $h : \mathbb{B}^q \to D^I_{p,q}$. By hypothesis, $u \circ h$ is C^q-smooth on $\overline{\mathbb{B}^q}$. By Graham's Theorem (Theorem 1), u is pluriharmonic in every $(1,q)-$space. But every minimal disk of $D^I_{p,q}$ is contained in some $(1,q)-$space. Thus u is harmonic on every minimal disk and by Theorem 5, u is pluriharmonic in $D^I_{p,q}$. This proves Theorem 2.

Moreover, note the boundary of a $(1,q)-$subspace intersects only with the smooth part E_1 of $\partial D^I_{p,q}$. Thus the above proof indeed leads to the following statement. Let $p \le q$ and $q \ge 2$.

Let $u \in C(\overline{D^I_{p,q}}) \cap C^q(D^I_{p,q})$ be Bergman-harmonic in $D^I_{p,q}$. Assume u extends C^q-smoothly across every smooth boundary point. Then u is pluriharmonic in $D^I_{p,q}$.

Using this statement and the proof of Theorem 3, we arrive at the following stronger version of Theorem 3. Assume $p \le q$.

Fix $0 \le k \le p-1$ and let $l := \max\{2, q-k\}$. Let $u \in C(\overline{D^I_{p,q}}) \cap C^l(D^I_{p,q})$ be Bergman-harmonic in $D^I_{p,q}$. Assume u extends C^l-smoothly across every point in $E_1 \cup E_2 \cdots \cup E_{k+1}$. Then u is pluriharmonic on every germ of complex submanifold in E_k.

An interesting question asks whether the assumption $u \in C(\overline{D^I_{p,q}})$ can be removed in the above statements.

References

[ABC] P. Ahern, J. Bruna, C. Cascante, H^p-theory for generalized M-harmonic functions in the unit ball, *Indiana Univ. Math. J.* **45** (1996), 103-135.

[CL] R. Chen and S. Li, Graham type theorem on bounded symmetric domains, arxiv: 1708.03434.

[D] J. D'Angelo, Several complex variables and the geometry of real hypersurfaces, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1993. xiv+272 pp. ISBN: 0-8493-8272-6.

[FHX] H. Fang, X. Huang and M. Xiao, Volume-preserving maps between Hermitian symmetric spaces of compact type, submitted, an expanded version at arxiv: 1602.01900v3.

[G1] C. Graham, The dirichlet problem for the bergman laplacian, **1** *Comm. in PDE* **8** (1983), 433-476.

[G2] C. Graham, The dirichlet problem for the bergman laplacian, **2** *Comm. in PDE* **8** (1983), 563-641.

[GL] C. Graham and J. Lee, Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains, Duke Math. J. **57** (1988), 697-720.

[H] L. K. Hua, Harmonic analysis of functions of several complex variables in the classical domains, (translated from the Russian, which was a translation of the Chinese original) Translations of Mathematical Monographs, 6. American Mathematical Society, Providence, R.I., 1979. iv+186 pp. ISBN: 0-8218-1556-3.

[HM] J.-M. Hwang and N. Mok, Rigidity of irreducible Hermitian symmetric spaces of the compact type under Khler deformation, *Invent. Math.*, **131** (1998), 393418.

[LN] S. Li and L. Ni, On the holomorphicity of proper harmonic maps between unit balls with the Bergman metrics, *Math. Ann.* **316** (2000) 333-354.

[LS] S. Li and E. Simon, Boundary behavior of harmonic functions in metrics of Bergman types on the polydisc, *Amer. J. Math.* **124** (2002), 1045-1057.

[LW] S. Li and D. Wei, On the rigidity theorem for harmonic functions in Kähler metric of Bergman type, *Sci. China Math.* **53** (2010), 779-790.

[Lu] Q.-K. Lu, Classical manifolds and Classical domains, Shanghai Scientific and Technical Press, Shanghai (1963).

[M] N. Mok, Metric rigidity theorems on Hermitian locally symmetric manifolds, Series in Pure Mathematics. 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989. xiv+278 pp.

[MT] N. Mok and I.-H. Tsai, Rigidity of convex realizations of irreducible bounded symmetric domains of rank ≥ 2, *J. Reine. Angew. Math.* **431** (1992), 91-122.

[Ng] S. Ng, Cycle spaces of flag domains on Grassmannians and rigidity of holomorphic mappings, *Math. Res. Lett.* **19** (2012), 1219-1236.

[PS] I.I. Pyatetskii-Shapiro, Automorphic functions and the geometry of classical domains, Mathematics and its applications, Vol. 8, Gordon and Breach Science Publishers, 1969.

[Wo] J. A. Wolf, Fine structures of Hermitian symmetric spaces, in Geometry of Symmetric Spaces, ed. Boothby-Weiss, Marcel-Dekker, New York 1972, pp.271-357.

[X] M. Xiao, A high order Hopf lemma for mappings into classical domains and applications, *submitted*, math.ucsd.edu/∼m3xiao/hopf-lemma.pdf.

Department of Mathematics, University of California San Diego, La Jolla, CA 92093, USA.

E-mail address: m3xiao@ucsd.edu